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Self-similarity and optical kinks in resonant nonlinear media

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We show that self-similar optical waves with a kink structure exist in a wide class of resonant nonlinear media, adequately treated in the two-level approximation. The self-similar structure of the present kinks is reflected in the time evolution of the field profile, atomic dipole moment, and one-atom inversion. We develop an analytical theory of such kinks. We show that the discovered kinks are accelerating nonlinear waves, asymptotically attaining their shape and the speed of light. We also numerically explore the formation and eventual disintegration of our kinks due to energy relaxation processes. Thus, the present kinks can be viewed as intermediate asymptotics of the system.

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The quest for structurally stable nonlinear waves, which maintain their intensity—and sometimes even phase—profiles in open physical, chemical, and biological systems, has been in the forefront of contemporary nonlinear science. Self-similar waves or similaritons—whose intensity profiles remain scaled replicas of themselves on propagation—can serve as a prominent example of structurally stable waves in open systems. Although self-similarity has long become textbook material in fluid and gas [1,2] and solid mechanics [2] as well as in plasma physics [3], the concept has only relatively recently percolated into nonlinear optics [4]. To date, self-similarity in Bragg gratings [5], stimulated Raman scattering [6], self-written waveguides [7], and fractal formation in nonlinear media [8] have been studied among other topics. More recently, however, the focus has shifted to two major classes of similaritons: asymptotic and solitonlike ones. The asymptotic temporal [9,10] or spatiotemporal [11] similaritons, forming in fiber amplifiers in the normal dispersion regime over long propagation distances, were theoretically predicted [9,10] and experimentally realized [10]. At the same time, bright and dark solitonlike similaritons have been theoretically investigated in fiber [12,13] and graded-index waveguide amplifiers [14], and in trapped Bose-Einstein condensates [15]. The vast majority of research on optical similaritons has, however, been aimed at understanding their behavior sufficiently far away from any internal resonance of the medium.

In this Rapid Communication, we show that an altogether present kind of optical similaritons, kink-like similaritons, can be supported by resonant nonlinear media. The different self-similar kinks accelerate, asymptotically acquiring the speed of light. An initial stage of their fast self-steepening is followed by asymptotically slow formation of quasi-steady-state kinks. All these features make the new kinks markedly different from the previously examined steady-state ones, Raman induced in optical fibers away from any resonance [16,17]. Interestingly, there is a direct analogy between the new optical kinks and the corresponding shocks in gas dynamics. The transverse relaxation processes, which determine the temporal width of the kink, are counterparts of gas viscosity. At the same time, the longitudinal relaxation processes, which lead to kink decay and eventual disintegration, are direct analogs of thermal processes in gases which cause shock disappearance. There are, however, two fundamental differences between optical kinks and shocks in gases or fluids. First, while the former do not require any background intensity, the latter form against a finite velocity background, determining the sound velocity in a gas or fluid. Second, whereas the gas shocks propagate with a constant supersonic speed, the present kinks are accelerating self-similar waves.

In this work, we treat resonant media in the two-level approximation. The model is sufficiently general to describe a broad range of resonant nonlinear media from atomic vapors and solids, doped with resonant atoms [18], to bulk semiconductors, doped with quantum dots [19]. Thus our analytical and numerical results may find applications to a multitude of physical systems as diverse as dilute gases and solids.

We begin by considering a light pulse with a carrier frequency ω near optical resonance frequency ω_0 of a two-level atom medium. To focus on the main aspects of the problem, we make two assumptions. First, we assume that the pulse is not chirped, implying that $\Omega^* = \Omega$. Second, we assume that the transverse relaxation rate $\gamma_\perp$—defined as the corresponding inverse relaxation time—dwarfs the longitudinal one, $\gamma_\parallel$, as well as the characteristic width of inhomogeneous broadening δ,

$$\gamma_\perp \gg \delta, \quad \gamma_\perp \gg \gamma_\parallel. \tag{1}$$

The first inequality implies that all impurity atoms are assumed to be effectively on resonance with the field such that inhomogeneous broadening can be ignored. The second inequality means that the atomic dipole moments evolve much faster than the atomic population dynamics unfolds. The existence of a hierarchy of widely separated in-time relaxation processes results in the emergence of two widely separated in-space characteristic propagation distances: a typical distance ζ_* over which the new kinks are formed and a characteristic energy relaxation distance ζ_{**} beyond which the kinks gradually decay. The different kinks maintain their self-similar structure in the intermediate range, $\zeta_* \ll \zeta \ll \zeta_{**}$.

Within the framework of our model and subject to the slowly varying envelope approximation (SVEA), the pulse evolution is governed by the reduced wave equation in the form,

$$\partial_\zeta \Omega = \frac{\omega N |d_{eg}|^2}{c\epsilon_0 \hbar} v. \tag{2}$$

Here, $\Omega = 2d_{eg}\mathcal{E}/\hbar$ is the Rabi frequency associated with the pulse amplitude $\mathcal{E}$, N is a density of impurity atoms, and d_{eg} is a dipole matrix element between the ground and excited states of any atom; the two relevant atomic states are appropriately labeled with the indices g and e. Furthermore, Eq. (2) is written in terms of the transformed coordinate and time, $\zeta = z$ and $\tau = t - z/c$. The relevant atomic dipole moment v and one-atom inversion w obey the Bloch equations [18] which, in our case, are simplified as

$$\partial_\tau v = -\gamma_\perp v + \Omega w, \tag{3}$$

and

$$\partial_\tau w = -\Omega v. \tag{4}$$

In deriving Eqs. (3) and (4), we neglected longitudinal relaxation processes, an assumption to be examined later with the help of numerical simulations.

The inspection of Eqs. (2)–(4) reveals the existence of self-similar solutions for the Rabi frequency,

$$\Omega(\tau,\zeta) = \gamma_\perp \overline{\Omega}(\eta), \tag{5}$$

and for the atomic variables,

$$v(\tau,\zeta) = e^{-\gamma_\perp \tau}\overline{v}(\eta), \quad w(\tau,\zeta) = e^{-\gamma_\perp \tau}\overline{w}(\eta). \tag{6}$$

Here the similarity variable is defined by the expression,

$$\eta = \alpha\zeta e^{-\gamma_\perp \tau}; \quad \alpha = \frac{2kN|d_{eg}|^2}{\gamma_\perp \epsilon_0 \hbar}, \tag{7}$$

where we introduced a linear absorption coefficient α and $k = \omega/c$.

The dimensionless Rabi frequency $\overline{\Omega}$ and scaled atomic variables, $\overline{v}$ and $\overline{w}$, obey the set of ordinary differential equations (ODE):

$$2\overline{\Omega}' = \overline{v}, \tag{8}$$

$$\eta\overline{v}' = -\overline{\Omega}\,\overline{w}, \tag{9}$$

and

$$(\eta\overline{w})' = \overline{\Omega}\,\overline{v}, \tag{10}$$

where the prime denotes a derivative with respect to the similarity variable. Combining Eqs. (8)–(10) and integrating once with the aid of the asymptotic condition $\overline{\Omega}(0) = \overline{\Omega}_\infty$, we arrive at the ODE for a kink profile,

$$\eta^2\overline{\Omega}'' = -\tfrac{1}{2}\overline{\Omega}\,(\overline{\Omega}^2 - \overline{\Omega}_\infty^2). \tag{11}$$

The analysis of Eq. (11) indicates that at the trailing edge of the pulse, $\tau \to +\infty$, the kink profile at any propagation distance asymptotically behaves as

$$\overline{\Omega} = \overline{\Omega}_\infty - |C|\eta^s; \quad s = \frac{1 + \sqrt{1 - 4\overline{\Omega}_\infty^2}}{2}, \tag{12}$$

where C is a constant. By the same token, at the leading edge, $\tau \to -\infty$, the kink field strength falls off as

$$\overline{\Omega} \sim \eta^{-q}, \quad q = \frac{-1 + \sqrt{1 + 2\overline{\Omega}_\infty^2}}{2}. \tag{13}$$

The kink profile is exhibited in Fig. 1 as a function of time. It follows from (12) and (13) that (i) the kink structure is determined entirely by the magnitude of the Rabi frequency jump, Ω_∞, and (ii) the kink has no chirp if the latter satisfies the inequality,

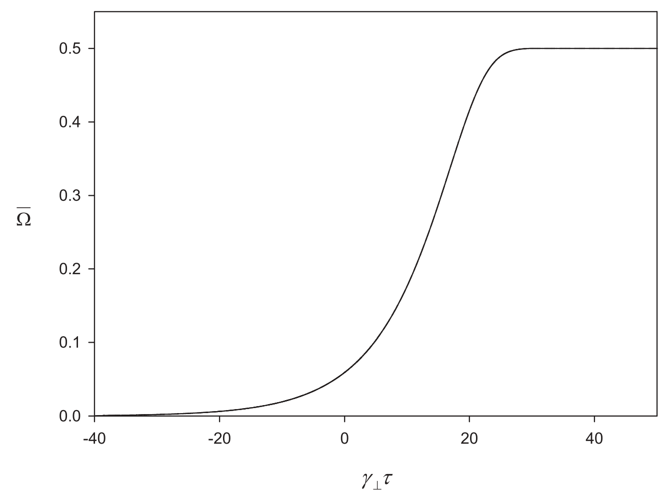

FIG. 1. Kink profile as a function of time, measured in the units of the transverse relaxation time, $T_\perp = 1/\gamma_\perp$. The dimensionless Rabi frequency jump is taken to be $\overline{\Omega}_\infty = 1/2$.

$$\Omega_\infty \leqslant \gamma_\perp/2; \tag{14}$$

otherwise our solution is not consistent. The condition (14) specifies the range of parameters for which kinks with monotonous profiles are realized in resonant media. It can be physically interpreted as follows. The Rabi frequency jump must be smaller than a certain critical value determined by the transverse damping constant such that the system is in an overdamped regime with no Rabi oscillations. The latter would lead to pulse chirping which, in turn, would cause modulations of the kink profile.

Further, we can infer from Eqs. (8)–(10) that the one-atom inversion can be expressed as

$$w(\zeta,\tau) = \frac{1}{\alpha\zeta}[\overline{\Omega}^2(\eta) - \overline{\Omega}_\infty^2]. \tag{15}$$

It follows at once from Eq. (15) and the definition of the inversion that at the leading edge of the kink: $w_\infty = -\overline{\Omega}_\infty^2/\zeta \geqslant -1$, implying that our self-similar solution is valid over the distances such that

$$\zeta \geqslant \zeta_* = \frac{\Omega_\infty^2}{\alpha\gamma_\perp^2}. \tag{16}$$

Here ζ_* is the lower bound of a characteristic distance over which the kink is formed. Thus the present kinklike similaritons are intermediate asymptotics of the system in the spirit of Ref. [2]. On the one hand, they form over distances of the order of ζ_*, after the transient dynamics, induced by specific initial conditions, have died away. On the other hand, the new kinks remain intact only over spatial scales much shorter than the characteristic energy relaxation distance determined by the longitudinal relaxation constant $\gamma_\parallel$.

We also note that at any (finite) propagation distance over which our kinks have already formed, the atomic dipole moment v asymptotically tends to zero, albeit asymmetrically, at both ends of the kink: $v \sim -e^{q\gamma_\perp\tau}$ at the leading edge, and

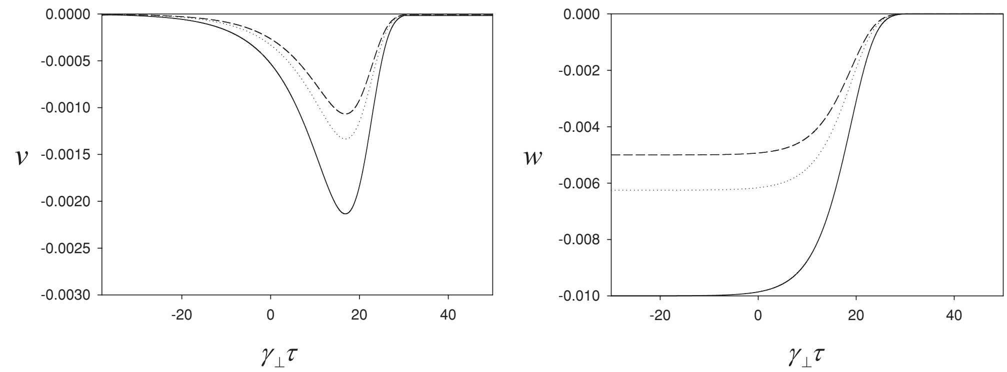

FIG. 2. Atomic dipole moment (left) and one-atom inversion (right) as functions of dimensionless time, $\gamma_\perp\tau$, displayed at several propagation distances: Solid, $\alpha\zeta = 25$; dotted, $\alpha\zeta = 40$, and dashed, $\alpha\zeta = 50$. The propagation distances are measured in the units of inverse Beer's absorption length, α^{-1}.

$v \sim -e^{-s\gamma_\perp\tau}$ at the trailing edge of the pulse, respectively. The time evolution of w and v is displayed in Fig. 2 for several propagation distances; a self-similar structure of the atomic state evolution is unambiguously reflected in the figure. Physically, the behavior of the inversion can be explained by observing that at the leading edge of the kink, where the light intensity is very small, there are much more atoms in their ground states than are excited atoms, resulting in a negative value of w. At the trailing edge, however, a large pulse amplitude saturates the medium, implying zero inversion.

Next, we reveal unusual dynamic properties of the discovered kinks. It follows from Eq. (7) that the speed U of a kink wave front depends on the propagation distance according to

$$U(\zeta) = \frac{\gamma_\perp \zeta c}{c + \gamma_\perp \zeta}. \tag{17}$$

Thus the discovered kinks accelerate on propagation, asymptotically attaining the speed of light. In reality, however, a characteristic distance over which their speed becomes sufficiently close to c, can be quite short, of the order of $\zeta_\infty \simeq c/\gamma_\perp$. For example, for solids or semiconductors doped with resonant atomic impurities or quantum dots, $10^{11} \leqslant \gamma_\perp \leqslant 10^{13}$, s^{-1}, leading to the estimate, $0.03 \leqslant \zeta_\infty \leqslant 3$ cm.

We now discuss kink formation. A constant background intensity at the trailing edge of the input wave is required to produce a kink. In laboratory, such pulses can be generated by switching on cw lasers, for example. In our numerical simulations, we then consider an adiabatically switched cw wave of the form,

$$\Omega(0,t) = \frac{\Omega_0}{1 + e^{-t/\tau_p}}, \tag{18}$$

where Ω_0 is the amplitude—measured in frequency units—of the cw laser field and τ_p is a characteristic time constant of the switching process. We emphasize that a particular functional form (18) is not important: we obtained qualitatively similar results for wave self-steepening and kink formation with different input wave profiles having a finite background intensity at the trailing edge.

The atoms are assumed to be initially in their ground states and the one-atom inversion obeys the Bloch equation,

$$\partial_\tau w = -\gamma_\parallel (w + 1) - \Omega v, \tag{19}$$

where the energy (longitudinal) relaxation processes are taken into account. Our numerical simulations indicate that monotonous kinks form provided that

$$\Omega_c \leqslant \Omega_0 \leqslant \gamma_\perp/2, \tag{20}$$

where the magnitude of a critical amplitude Ω_c depends on the value of $\gamma_\parallel$. The presence of a critical power threshold for kink formation is explained as follows. The incident wave should have enough power to start self-steepening despite energy losses caused by longitudinal relaxation processes. Clearly, the shorter the longitudinal relaxation time, the greater the initial amplitude is required to generate a kink.

The results of numerical simulations of Eqs. (2), (3), and (19), with the initial condition (18), are displayed in Figs. 3 and 4. In Fig. 3, we show self-similar kink formation for sufficiently long energy relaxation times, $T_\parallel/T_\perp = 10^4$. After a brief stage of fast self-steepening, exhibited in the inset to

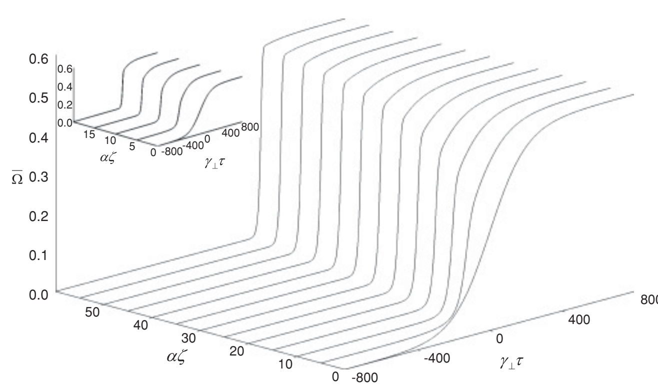

FIG. 3. Dimensionless Rabi frequency $\overline{\Omega}$ of a forming kink as a function of dimensionless time, $\gamma_\perp\tau$, and propagation distance $\alpha\zeta$. The ratio of transverse to longitudinal relaxation times is $T_\perp/T_\parallel = 10^{-4}$. The initial parameters are $\Omega_0 = 0.5\gamma_\perp$ and $\tau_p = 100T_\perp$. The inset shows the initial stage of fast self-steepening.

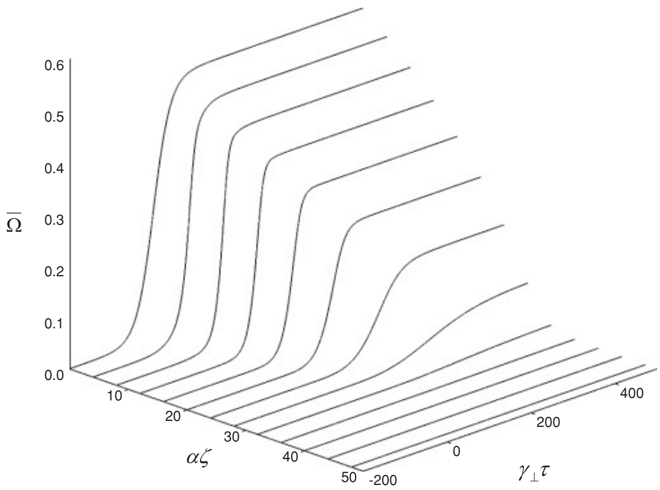

FIG. 4. Dimensionless Rabi frequency $\overline{\Omega}$ of a forming kink as a function of dimensionless time, $\gamma_\perp\tau$, and propagation distance $\alpha\zeta$. The ratio of transverse to longitudinal relaxation times is $T_\perp/T_\parallel = 10^{-2}$. The initial parameters are $\Omega_0 = 0.5\,\gamma_\perp$ and $\tau_p = 20\,T_\perp$.

the figure, the wave self-steepening slows down at distances of the order of ζ_∞, corresponding to the kink having attained the speed close to the speed of light. The subsequent asymptotic self-steepening leads to quasi-steady-state kink formation. The critical amplitude in this case is found to be $\Omega_c = 0.2\gamma_\perp$. The numerically obtained kink profile coincides, to a good accuracy, with the analytically determined one, which justifies neglecting the longitudinal relaxation processes in Eq. (4) [20]. In Fig. 4, we exhibit the influence of the latter on kink formation. One can see in the figure that for relatively short energy relaxation times—in our instance for $T_\parallel/T_\perp = 10^2$—the emerging kinks survive only briefly: the energy dissipation eventually takes its toll over longer distances.

Finally, we briefly mention the systems in which present kinks can be realized. The characteristic transverse and longitudinal relaxation times for solids, doped with resonant atom impurities, fall into the ranges $10^{-6} \leqslant T_\parallel \leqslant 10^{-3}$ and $10^{-13} \leqslant T_\perp \leqslant 10^{-11}$ s [21,22], respectively. Thus, $10^5 \leqslant \gamma_\perp/\gamma_\parallel \leqslant 10^{10}$, which makes solids ideal for realization of the present kinks, provided inhomogeneous broadening can be reduced by preparing clean enough samples. At the same time, relaxation times for bulk semiconductors, doped with quantum dots, range as follows, $10^{-12} \leqslant T_\parallel \leqslant 10^{-4}$ and $10^{-13} \leqslant T_\perp \leqslant 10^{-12}$ s [21]. Consequently, $1 \leqslant \gamma_\perp/\gamma_\parallel \leqslant 10^9$, and hence our kinks can be realized in some semiconductor systems as well.

In conclusion, we have discovered and analytically described a class of self-similar waves in resonant nonlinear media, optical kinks. The present kinks can form in two-level media under the assumption that the longitudinal relaxation time is much longer than the transverse one. Thus a wide range of intermediate propagation distances exists over which the kinks are formed as a result of the interplay of optical nonlinearity and the phase (transverse) relaxation processes; yet the influence of the energy (longitudinal) relaxation processes is still negligible. We stress, however, that our results pertain to the case of negligible inhomogeneous broadening, which requires rather clean samples, and for chirp-free waves. We conjecture that the presence of a chirp may lead to oscillatory kink profiles.

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